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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIRST SEMESTER EXAMINATION, DECEMBER 2016

FIRST YEAR [BATCH 2016-19]

Date : 12/12/2016 **MATHEMATICS [Honours]**

Time : 11 am – 3 pm **Paper : I** Full Marks : 100

[Use a separate Answer Book for each Group]

Group – A

Unit – I

(***Answer any five questions***) [5×5]

1. a) For any three sets A, B, C prove that $A\times(B\cap C)=(A\times B)\cap(A\times C)$. [2]

   b) Prove that union of two symmetric relations on a set is a symmetric relation. Give an example to prove that it is not true for transitive relation. [2+1]

2. a) Let $f:A\to B$ be an onto mapping and P,Q be subsets of B. Prove that $f^{-1}(P\cap Q)=f^{-1}(P)\cap f^{-1}(Q)$. [3]

   b) Define a bijective map from $\mathbb{Q}$ to $\mathbb{Q}\cup\{\sqrt{2}\}$. [2]

3. a) Let X be a nonempty set. Prove that there does not exist any surjective map from X to P(X), the power set of X. [3]

   b) Prove that the relation ρ on $\mathbb{R}$ defined by '$x\rho y$' iff $x-y\in\mathbb{Q}(x,y\in\mathbb{R})$ is an equivalence relation. Find the equivalence class containing the element '0'. [2]

4. a) Let S be a finite set containing at least two elements. Prove that the number of all even permutations on S is equal to the number of all odd permutations on it. [3]

   b) Let a = (1 6 3 7)(2 4 5) be an element of S_7. Is a^{-1} an even permutation? [2]

5. Define the centre of a group. If G is a noncommutative group of order 10 then show that G has a trivial centre. [1+4]

6. a) In a group (G, ·), $a^3 = e$ (the identity of G) and $a\cdot b\cdot a^{-1}=b^2(a,b\in G)$; find the order of b if $b\neq e$. [2]

   b) If (H, ·) be a subgroup of a group (G, ·), prove that the two left cosets of H in G are either disjoint or identical. [3]

7. Prove that every subgroup of a cyclic group is cyclic. [5]

8. a) Let G be a finite group and H, K are subgroups of G such that $K\subseteq H\subseteq G$. Prove that $[G:K]=[G:H]\cdot[H:K]$. [3]

   b) Find all subgroups of the group $(Z_{10},+)$. [2]

Unit – II

(***Answer any five questions***) [5×5]

9. If $x,y\in\mathbb{R}$ with y > 0, then prove that $\exists n\in\mathbb{N}$ such that ny > x. Hence show that $\forall\ y>0\ \exists\ n\in\mathbb{N}$ such that $0<\frac{1}{n}<y$. [4+1]

10. a) Prove that for any two real numbers x and y with x < y, there exists a rational number r such that x < r < y. [3]

b) Prove that $\text{Sup}\{r \in Q : r < b\} = b$ for any real number b where $\mathbb{Q}$ denotes the set of rational numbers. [2]

11. Prove that every infinite bounded subset in $\mathbb{R}$ has at least one limit point in $\mathbb{R}$. [5]

12. a) Let S be a nonempty subset of $\mathbb{R}$ which is bounded above. If M = Sup S and $M \notin S$ then prove that M is a limit point of S. [2]

b) Let S and T be two bounded set of real numbers and $X = \{a - b : a \in S, b \in T\}$. Prove that Sup X = Sup S – Inf T. [3]

13. Show that the interval (0,1) in $\mathbb{R}$ is not denumerable. [5]

14. Prove that the sequence $\{u_n\}_n$ defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ $\forall n \in \mathbb{N}$ converges to 2. [5]

15. a) If the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ of a sequence $\{u_n\}$ converge to the same limit ℓ then prove that the sequence $\{u_n\}$ is convergent and $\lim u_n = \ell$. [3]

b) Give an example of an unbounded sequence with two subsequences one of which is convergent and the other divergent. [2]

16. Let $S \subseteq \mathbb{R}$ and f, g, h are functions from S to $\mathbb{R}$ such that $f(x) < g(x) < h(x) \forall x \in S - \{a\}$ where $a \in S'$ (derived set of S) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = \ell$. Show that $\lim_{x \to a} g(x) = \ell$. [5]

<u>Group – B</u>

<u>Unit – III</u>

(*Answer <u>any three</u> questions*) [3×5]

17. Prove that the orthocentre (p,q) of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $\ell x + my = 1$ is given by $\frac{p}{\ell} = \frac{q}{m} = \frac{a+b}{am^2 - 2h\ell m + b\ell^2}$. [5]

18. Reduce the equation $4x^2 - 4xy + y^2 - 8x - 6y + 5 = 0$ to its cannonical form and state the type of the conic. [5]

19. Find the equation of the circle which passes through the focus of the conic $\frac{\ell}{r} = 1 - e\cos\theta$ and touches it at the point $\theta = \alpha$. [5]

20. Find the locus of the poles of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which subtend right angles at the centre. [5]

21. If circles be described on two semi-conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as diameters, prove that the locus of the second point of intersection is $2(x^2 + y^2)^2 = a^2x^2 + b^2y^2$. [5]

<u>Unit – IV</u>

(*Answer <u>any two</u> questions*) [2×5]

22. Show that the necessary and sufficient condition of collinearity of three points A, B, C is that there exist scalars x, y, z not all zero, such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, x + y + z = 0, where $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of A, B, C respectively with respect to some origin. [5]

23. Prove that any vector $\vec{r}$ can be expressed as a linear combination of three noncoplanar vectors $\vec{a}, \vec{b}, \vec{c}$ in the form $\vec{r} = \frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{a} + \frac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{b} + \frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{c}$. [5]

24. Find the equation of the plane which contains the line $\vec{r} = 2\hat{i} + t(\hat{j} - \hat{k})$ and perpendicular to the plane $\vec{r} \cdot (\hat{i} + \hat{k}) = 3$. Find the point where this plane meets the line $\vec{r} = t(2\hat{i} + 3\hat{j} + \hat{k})$. [4+1]

<u>Unit – V</u>

(*Answer <u>any five</u> questions*) [5×5]

25. Obtain the differential equation corresponding to the primitive $(x-a)^2 + (y-b)^2 = r^2$, where r is a fixed constant and a, b are arbitrary constants. Give a geometrical interpretation of the result. [4+1]

26. If $\frac{dy}{dx} = f(ax + by + c)$, show that the substitution ax + by + c = v will change it to a separable equation. Use it to solve $\frac{dy}{dx} + 2xy = x^2 + y^2$. [2+3]

27. Reduce the equation $xp^2 - 2yp + x + 2y = 0$ to Clairaut's form by using the substitution $x^2 = u$ and y – x = v and hence solve it, where $p \equiv \frac{dy}{dx}$. [5]

28. Find the orthogonal trajectories of the cardioides $r = a(1 - \cos\theta)$, ‘a’ being a variable parameter. [5]

29. Solve : $\frac{d^2y}{dx^2} + 4y = H(x)$, where $H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$, given that y(0) = 0, $\frac{dy}{dx} = 1$ at y = 0 and y, $\frac{dy}{dx}$ are continuous at x = 0. [5]

30. Solve the equation $\frac{d^2y}{dx^2} + 4y = x^2 \sin 2x$ by method of undetermined coefficients. [5]

31. Using the method of variation of parameters, solve the differential equation $(x-1)D^2y - xDy + y = (x-1)^2$, where $D \equiv \frac{d}{dx}$. [5]

32. By the use of symbolic operator , find the solution of the equation
$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$$ [5]

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